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INFLUENCE OF LATERAL INERTIA ON CRACK PROPAGATION IN PLATES

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Abstract—The implications of lateral inertia on linear elastodynamic crack propagation in plates are studied, using a more accurate approximation of the equations of motion than the one of plane stress. It is found that the stress–strain field near the crack edge is, in general, *plane stress like* in the respect that the energy release rate coincides with what is predicted from the plane stress approximation. The stress intensity factor and the crack opening displacement are, however, different from those found from this approximation, and they are obviously governed essentially by plane strain domination near the crack edge and to a lesser extent by lateral inertia.

1. INTRODUCTION

The vast majority of crack problems appear to concern plane cases, either plane strain or plane stress. Pure plane strain cases can, however, seldom be realized, and "plane stress" usually refers to the plane stress approximation generally used when dealing with "thin" plates. It is, however, well known that a crack in a plate is subjected to something resembling plane strain conditions near the crack edge but which comes close to the plane stress approximation far away from the edge. Assuming infinitesimally small scale yielding, Yang and Freund (1985) and Rosakis and Ravi-Chandar (1986) showed that, for stationary cracks, the plane stress approximation appears to hold with decent accuracy at distances from the crack edge that are larger than about half the plate thickness. Broberg (1987) showed that although different conditions prevail near to and far away from the crack edge, the J integral for a path near the edge equals the J integral for a sufficiently remote path (note that the J integral is not defined for paths in the intermediate mixed plane strain/plane stress region).

For running cracks in a plate an additional complication enters, namely lateral inertia, the implications of which will be investigated. To this end a more accurate approximation of the equations of motion than the plane stress approximation is sought, and this more accurate approximation is then used for analysing a simple problem of dynamic crack propagation in a plate.

2. EQUATIONS OF MOTION

Study a plate with thickness $2h$, subjected to in-plane loading at the edges. A Cartesian coordinate system, x, y, z, is introduced with the origin in the mid-plane and the z direction normal to the plate surfaces. Conventional denotations, u, v, w for displacements, and ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{zx} for strains, are used.

It is now assumed that an essentially in-plane motion in the plate can be described approximately by an equation of motion that contains only x, y and t as independent variables, as in plane stress or plane strain cases. Here, however, the influence of lateral inertia is also considered. It is assumed that

$$u = u(x, y, t) \tag{1}$$

$$v = v(x, y, t) \tag{2}$$

$$\varepsilon_z = \varepsilon_z(x, y, t) \tag{3}$$

and, thus

$$w = \varepsilon_z \cdot z. \tag{4}$$

Then

$$\varepsilon_x = \varepsilon_x(x, y, t) \tag{5}$$

$$\varepsilon_y = \varepsilon_y(x, y, t) \tag{6}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{7}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} = z\frac{\partial \varepsilon_z}{\partial x} \tag{8}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} = z\frac{\partial \varepsilon_z}{\partial y}. \tag{9}$$

The equations of motion will now be determined by using Hamilton's principle. To this end the kinetic energy, T, and the potential energy, U, are needed. One obtains:

$$T = \frac{\rho}{2}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) = \frac{\rho}{2}(\dot{u}^2 + \dot{v}^2 + z^2\dot{\varepsilon}_z^2) \tag{10}$$

$$U = \tfrac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}). \tag{11}$$

After use of Hooke's law

$$\sigma_x = 2\mu\left[\varepsilon_x + \frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right], \text{etc.} \tag{12}$$

$$\tau_{xy} = \mu\gamma_{xy}, \text{etc.} \tag{13}$$

where μ is the modulus of rigidity and ν Poisson's ratio, one obtains

$$T - U = \frac{\rho}{2}(\dot{u}^2 + \dot{v}^2 + z^2\dot{\varepsilon}_z^2) - \frac{\mu}{1-2\nu}[(1-\nu)(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2) + 2\nu(\varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x)] - \frac{\mu}{2}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2). \tag{14}$$

The variation

$$\delta I = \delta \int_t \int_x \int_y \int_{-h}^{h} (T - U)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x\, \mathrm{d}t = 0. \tag{15}$$

Here the subscripts t, x and y indicate arbitrary regions of integration in the t and xy domains. The inner integral is now written as

$$\int_{-h}^{h}(T-U)\,\mathrm{d}z = F\left(\dot{u},\dot{v},\dot{\varepsilon}_z,\varepsilon_x,\varepsilon_y,\varepsilon_z,\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}\cdot\frac{\partial \varepsilon_z}{\partial x},\frac{\partial \varepsilon_z}{\partial y}\right)$$

$$= \rho h\left[\dot{u}^2+\dot{v}^2+\frac{h^2}{3}\dot{\varepsilon}_z^2\right]$$

$$-\frac{2\mu h}{1-2\nu}[(1-\nu)(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2)+2\nu(\varepsilon_x\varepsilon_y+\varepsilon_y\varepsilon_z+\varepsilon_z\varepsilon_x)]$$

$$-\mu h\left\{\left(\frac{\partial u}{\partial y}\right)^2+2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}+\left(\frac{\partial v}{\partial x}\right)^2+\frac{h^2}{3}\left[\left(\frac{\partial \varepsilon_z}{\partial x}\right)^2+\left(\frac{\partial \varepsilon_z}{\partial y}\right)^2\right]\right\}. \quad (16)$$

Variation gives:

$$\delta I = \int_t\int_x\int_y\left[\frac{\partial F}{\partial \dot{u}}\delta\dot{u}+\cdots\right]\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = 0, \quad (17)$$

where $\cdots$ indicates that variation is performed with respect to all arguments of function F. Partial integrations of all integrals, except

$$\int_t\int_x\int_y\frac{\partial F}{\partial \varepsilon_z}\delta\varepsilon_z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \quad (18)$$

which is already in the desired form, express δI in terms of variations of u, v and ε_z. Thus, for instance

$$\int_t\int_x\int_y\frac{\partial F}{\partial \dot{u}}\delta\dot{u}\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x = \int_y\int_x\left[\frac{\partial F}{\partial \dot{u}}\delta u\right]_t\mathrm{d}y\,\mathrm{d}x-\int_t\int_x\int_y\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{u}}\right)\delta u\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x$$

$$= -\int_t\int_x\int_y\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{u}}\right)\delta u\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x \quad (19)$$

and, hence, δI becomes expressed in the form

$$\delta I = \int_t\int_x\int_y\left\{\left[-\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{u}}\right)+\cdots\right]\delta u+[\cdots]\delta v+[\cdots]\delta\varepsilon_z\right\}\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x. \quad (20)$$

Since the variations of u, v and ε_z are arbitrary, except that they vanish at the domain boundaries, the variation of δI vanishes only if the expressions inside square brackets in eqn (20) vanish:

$$\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{u}}\right)+\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \varepsilon_x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial(\partial u/\partial y)}\right) = 0 \quad (21)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{v}}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial \varepsilon_y}\right)+\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial(\partial v/\partial x)}\right) = 0 \quad (22)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{\varepsilon}_z}\right)+\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial(\partial \varepsilon_z/\partial x)}\right)+\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial(\partial \varepsilon_z/\partial y)}\right)-\frac{\partial F}{\partial \varepsilon_z} = 0. \quad (23)$$

Insertion of F, from eqn (16), gives the equations of motion

$$\rho\frac{\partial^2 u}{\partial t^2}-\frac{2(1-\nu)\mu}{1-2\nu}\cdot\frac{\partial^2 u}{\partial x^2}-\frac{\mu}{1-2\nu}\cdot\frac{\partial^2 v}{\partial x\,\partial y}-\mu\frac{\partial^2 u}{\partial y^2}-\frac{2\nu\mu}{1-2\nu}\cdot\frac{\partial \varepsilon_z}{\partial x}=0 \tag{24}$$

$$\rho\frac{\partial^2 v}{\partial t^2}-\frac{2(1-\nu)\mu}{1-2\nu}\cdot\frac{\partial^2 v}{\partial y^2}-\frac{\mu}{1-2\nu}\cdot\frac{\partial^2 u}{\partial x\,\partial y}-\mu\frac{\partial^2 v}{\partial x^2}-\frac{2\nu\mu}{1-2\nu}\cdot\frac{\partial \varepsilon_z}{\partial y}=0 \tag{25}$$

$$\frac{\rho h^2}{3}\cdot\frac{\partial^2\varepsilon_z}{\partial t^2}+\frac{2(1-\nu)\mu}{1-2\nu}\varepsilon_z+\frac{2\nu\mu}{1-2\nu}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\mu h^2}{3}\left(\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}\right)=0. \tag{26}$$

Elimination of ε_z between eqns (24) and (25) gives

$$\left(\Delta-\frac{1}{c_s^2}\cdot\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)=0, \tag{27}$$

where Δ is the Laplace operator and c_s is the propagation velocity of S-waves (secondary waves), i.e.

$$c_s^2=\frac{\mu}{\rho}. \tag{28}$$

The displacement vector in the xy plane can be represented by the potentials Φ and Ψ according to Helmholtz's decomposition:

$$u=\frac{\partial\Phi}{\partial x}+\frac{\partial\Psi}{\partial y} \tag{29}$$

$$v=\frac{\partial\Phi}{\partial y}-\frac{\partial\Psi}{\partial x}. \tag{30}$$

In eqn (27) the displacements u and v appear only in the context of the rotation, $\partial u/\partial y-\partial v/\partial x=\Delta\Psi$, so that the equation can be written in the form

$$\Delta\left(\Delta\Psi-\frac{1}{c_s^2}\cdot\frac{\partial^2\Psi}{\partial t^2}\right)=0. \tag{31}$$

Multiplication of eqn (24) by $\partial/\partial x$ and of eqn (25) by $\partial/\partial y$ gives, after addition, an expression for $\Delta\varepsilon_z$, which is then used to eliminate ε_z from eqn (26). It turns out that u and v in the resulting equation appear only in the context of the in-plane dilatation, $\partial u/\partial x+\partial v/\partial y=\Delta\Phi$, and the equation can be written in the form

$$\Delta\left\{\Delta\Phi-\frac{1}{c_{pl}^2}\cdot\frac{\partial^2\Phi}{\partial t^2}-\frac{(1-\nu)h^2}{6}\left(\Delta-\frac{1}{c_s^2}\cdot\frac{\partial^2}{\partial t^2}\right)\left(\Delta\Phi-\frac{1}{c_p^2}\cdot\frac{\partial^2\Phi}{\partial t^2}\right)\right\}=0, \tag{32}$$

where c_p is the propagation velocity of P-waves (primary waves) in an infinite medium, and c_{pl} is the propagation velocity of P-waves in a plate under the plane stress approximation. It can be noted that the propagation of S-waves is the same for waves in a plate as for an infinite medium—they are simply equivoluminal waves. P-waves in an infinite medium are irrotational. The two P-wave velocities in eqn (32) are given by

$$c_p^2 = \frac{2(1-\nu)}{1-2\nu} \cdot \frac{\mu}{\rho} = c_s^2/k^2 \tag{33}$$

$$c_{pl}^2 = \frac{2}{1-\nu} \cdot \frac{\mu}{\rho} = 4(1-k^2)c_s^2, \tag{34}$$

where k, the ratio between equivoluminal and irrotational wave velocities, is related to Poisson's ratio ν through:

$$k^2 = \frac{1-2\nu}{2(1-\nu)}. \tag{35}$$

Equation (31) is the same as for plane stress or plane strain. Equation (32) agrees with the plane stress equation for Φ when $h \to 0$ and with the plane strain equation when $h \to \infty$. There is also one part in eqn (32) that indicates S-wave propagation coupled to P-waves—this might be explained by the fact that P-waves in a plate are not fully irrotational, since they will be accompanied by shear strains γ_{xz} and γ_{yz}, due to waves of lateral contraction.

In analogy with the plane stress or the plane strain case Φ and Ψ can be sought among functions that satisfy eqns (32) and (31) inside the braces, only. Thus one obtains the equations of motion

$$\Delta\Phi - \frac{1}{c_{pl}^2} \cdot \frac{\partial^2\Phi}{\partial t^2} - \frac{h^2}{12(1-k^2)}\left(\Delta - \frac{1}{c_s^2} \cdot \frac{\partial^2}{\partial t^2}\right)\left(\Delta\Phi - \frac{1}{c_p^2} \cdot \frac{\partial^2\Phi}{\partial t^2}\right) = 0 \tag{36}$$

$$\Delta\Psi - \frac{1}{c_s^2} \cdot \frac{\partial^2\Psi}{\partial t^2} = 0. \tag{37}$$

3. REPRESENTATION OF STRAINS AND STRESSES

From the representations (29) and (30) of displacements one obtains the strains

$$\varepsilon_x = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Psi}{\partial x\,\partial y} \tag{38}$$

$$\varepsilon_y = \frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x\,\partial y} \tag{39}$$

$$\gamma_{xy} = 2\frac{\partial^2\Phi}{\partial x\,\partial y} + \frac{\partial^2\Psi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x^2}. \tag{40}$$

The x and y derivatives of the third normal strain are found from eqns (24) and (25). Making use of eqn (37) one obtains after integration:

$$\varepsilon_z = \frac{1}{1-2k^2}\left(\frac{1}{c_p^2} \cdot \frac{\partial^2\Phi}{\partial t^2} - \Delta\Phi\right). \tag{41}$$

The stresses are now found by using Hooke's law:

$$\sigma_x = \mu\left(\frac{1}{c_s^2}\cdot\frac{\partial^2\Phi}{\partial t^2} - 2\frac{\partial^2\Phi}{\partial y^2} + 2\frac{\partial^2\Psi}{\partial x\,\partial y}\right) \tag{42}$$

$$\sigma_y = \mu\left(\frac{1}{c_s^2}\cdot\frac{\partial^2\Phi}{\partial t^2} - 2\frac{\partial^2\Phi}{\partial x^2} - 2\frac{\partial^2\Psi}{\partial x\,\partial y}\right) \tag{43}$$

$$\sigma_z = \frac{\mu}{1-2k^2}\left(\frac{1}{c_s^2}\cdot\frac{\partial^2\Phi}{\partial t^2} - \frac{c_s^2}{c_{pl}^2}\Delta\Phi\right) \tag{44}$$

$$\tau_{xy} = \mu\gamma_{xy} \tag{45}$$

$$\tau_{yz} = \mu z\frac{\partial\varepsilon_z}{\partial y} \tag{46}$$

$$\tau_{xz} = \mu z\frac{\partial\varepsilon_z}{\partial x}. \tag{47}$$

4. UNI-DIRECTIONAL WAVES

Since the equation for Ψ is the same as for plane strain, it is obvious that unidirectional S-waves in the plate travel with constant velocity c_s, i.e. there is no dispersion for S-wave pulses. The propagation velocity c of a harmonic P-wave, travelling in the x direction,

$$\Phi = A e^{i\omega(t-x/c)} \tag{48}$$

will, however, vary with ω. Insertion into eqn (36) gives the equation

$$1 - \frac{c^2}{c_{pl}^2} + \frac{\pi^2}{12(1-k^2)L^2}\left(1-\frac{c^2}{c_s^2}\right)\left(1-\frac{c^2}{c_p^2}\right) = 0, \tag{49}$$

where $L = \lambda/2h$ is the ratio between the wavelength $\lambda = 2\pi c/\omega$ and the plate thickness $2h$. The solution c/c_{pl} is plotted against L in Fig. 1 for Poisson's ratio $\nu = 1/3$, i.e. $k = 1/2$. It is seen that the propagation velocity increases from the one for S-waves, $c = c_s$, towards the

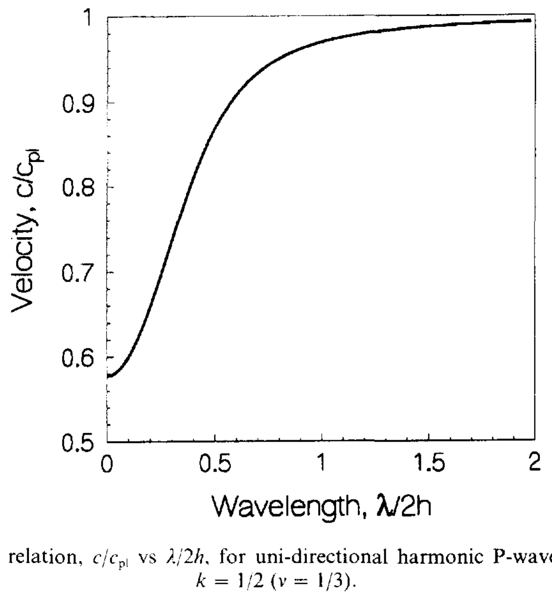

Fig. 1. Dispersion relation, c/c_{pl} vs $\lambda/2h$, for uni-directional harmonic P-waves $c \to c_s$ as $\lambda \to 0$; $k = 1/2$ ($\nu = 1/3$).

propagation velocity c_{pl} for plane stress P-waves, when the wavelength increases. This is perhaps the expected behaviour, but one should keep in mind the approximation that ε_z is taken to be independent of z, which implies that it is in general less good for wavelengths that are smaller than the plate thickness than for larger ones. The series expansion for large wavelengths is:

$$\frac{c}{c_{pl}} = 1-\pi^2\frac{(3-4k^2)(1-2k^2)^2}{24(1-k^2)}\cdot\left(\frac{2h}{\lambda}\right)^2 + \pi^4\frac{(3-4k^2)(1-2k^2)^2(13-112k^2+212k^4-112k^6)}{1152(1-k^2)^2}\cdot\left(\frac{2h}{\lambda}\right)^4+\cdots. \tag{50}$$

For $k = 1/2$ one obtains:

$$\frac{c}{c_{pl}} = 1-\frac{\pi^2}{36}\left(\frac{2h}{\lambda}\right)^2-\frac{7\pi^4}{2592}\left(\frac{2h}{\lambda}\right)^4+\cdots. \tag{51}$$

5. RAYLEIGH WAVES

Rayleigh waves propagating along a plate edge will exhibit dispersion, due to lateral inertia. A semi-infinite plate, $y \leqslant 0$, is studied with respect to the possibility of wave propagation along the edge. To this end a constant amplitude harmonic wave, with angular frequency ω, is assumed to travel with velocity c_R in the positive x direction. The boundary conditions on $y = 0$ are

$$\sigma_y = 0 \tag{52}$$

$$\tau_{xy} = 0 \tag{53}$$

$$\tau_{yz} = 0 \tag{54}$$

and, in addition, these stresses must vanish as $y \to -\infty$.

It is convenient to introduce dimensionless quantities,

$$X = \frac{x}{\chi},\quad Y = \frac{y}{\chi}, \tag{55}$$

$$T = \frac{c_s t}{\chi},\quad \Omega = \frac{\omega\chi}{c_s},\quad C_R = \frac{c_R}{c_s}, \tag{56}$$

where $\chi = \kappa h/\sqrt{3}$ with κ denoting the ratio between the propagation velocities of S- and P-waves in the plane stress approximation, i.e.

$$\kappa = \frac{c_s}{c_{pl}} = \frac{1}{2\sqrt{(1-k^2)}} = \sqrt{\left(\frac{1-\nu}{2}\right)}. \tag{57}$$

Keeping, for simplicity, the function symbols Φ and Ψ, although the arguments are now X, Y, T, the equations of motion read:

$$\Delta\Phi-\kappa^2\frac{\partial^2\Phi}{\partial T^2}-\left(\Delta-\frac{\partial^2}{\partial T^2}\right)\left(\Delta\Phi-k^2\frac{\partial^2\Phi}{\partial T^2}\right)=0 \tag{58}$$

$$\Delta\Psi-\frac{\partial^2\Psi}{\partial T^2}=0. \tag{59}$$

The boundary conditions (52) and (53) take the form:

$$\frac{\partial^2\Phi}{\partial T^2}-2\frac{\partial^2\Phi}{\partial X^2}-2\frac{\partial^2\Psi}{\partial X\,\partial Y}=0 \tag{60}$$

$$2\frac{\partial^2\Phi}{\partial X\,\partial Y}+\frac{\partial^2\Psi}{\partial Y^2}-\frac{\partial^2\Psi}{\partial X^2}=0. \tag{61}$$

Assume that

$$\Phi=f(Y)\cos\left[\Omega(T-X/C_{\mathrm{R}})\right] \tag{62}$$

$$\Psi=g(Y)\sin\left[\Omega(T-X/C_{\mathrm{R}})\right]. \tag{63}$$

(A cosine term for Ψ would disappear when the boundary conditions are satisfied.)
Insertion of eqns (62) and (63) into (58) and (59) gives

$$\frac{\mathrm{d}^4 f}{\mathrm{d}Y^4}-(1+b_{\mathrm{s}}+b_{\mathrm{p}})\frac{\mathrm{d}^2 f}{\mathrm{d}Y^2}+(b_{\mathrm{p}}b_{\mathrm{s}}+b_{\mathrm{pl}})f=0 \tag{64}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}Y^2}-b_{\mathrm{s}}g=0 \tag{65}$$

with the solutions

$$f=A\exp(\alpha_1 Y)+B\exp(\alpha_2 Y) \tag{66}$$

$$g=C\exp(\alpha_3 Y), \tag{67}$$

where

$$b_{\mathrm{s}}=\Omega^2\left(\frac{1}{C_{\mathrm{R}}^2}-1\right),\quad b_{\mathrm{p}}=\Omega^2\left(\frac{1}{C_{\mathrm{R}}^2}-k^2\right),\quad b_{\mathrm{pl}}=\Omega^2\left(\frac{1}{C_{\mathrm{R}}^2}-\kappa^2\right) \tag{68}$$

$$\alpha_1=\sqrt{\left\{\frac{1+b_{\mathrm{s}}+b_{\mathrm{p}}}{2}-\tfrac{1}{2}\left[(1+b_{\mathrm{s}}-b_{\mathrm{p}})^2+4(b_{\mathrm{p}}-b_{\mathrm{pl}})\right]^{1/2}\right\}} \tag{69}$$

$$\alpha_2=\sqrt{\left\{\frac{1+b_{\mathrm{s}}+b_{\mathrm{p}}}{2}+\tfrac{1}{2}\left[(1+b_{\mathrm{s}}-b_{\mathrm{p}})^2+4(b_{\mathrm{p}}-b_{\mathrm{pl}})\right]^{1/2}\right\}} \tag{70}$$

$$\alpha_3=\sqrt{b_{\mathrm{s}}}. \tag{71}$$

Use of the boundary conditions leads to the system of homogeneous equations

$$(2-C_R^2)(A+B)-2\sqrt{(1-C_R^2)}C = 0 \tag{72}$$

$$2C_R(\alpha_1 A+\alpha_2 B)-\Omega(2-C_R^2)C = 0 \tag{73}$$

$$\alpha_1[\alpha_1^2 C_R^2-\Omega^2(1-k^2C_R^2)]A+\alpha_2[\alpha_2^2C_R^2-\Omega^2(1-k^2C_R^2)]B = 0 \tag{74}$$

and the dispersion relation is found by putting the system determinant equal to zero:

$$\Omega(2-C_R^2)^2-\frac{4\alpha_1\alpha_2(\alpha_1+\alpha_2)C_R^3\sqrt{(1-C_R^2)}}{[(\alpha_1+\alpha_2)^2-\alpha_1\alpha_2]C_R^2-(1-k^2C_R^2)\Omega^2} = 0. \tag{75}$$

By putting $\Omega = 0$ one obtains

$$(2-C_R^2)^2-4\sqrt{(1-\kappa^2C_R^2)}\sqrt{(1-C_R^2)} = 0 \tag{76}$$

which is the equation for the velocity c_{Rpl} of Rayleigh waves along a plate edge in the plane stress approximation, and by letting $\Omega \to \infty$ one obtains the plane strain equation,

$$(2-C_R^2)^2-4\sqrt{(1-k^2C_R^2)}\sqrt{(1-C_R^2)} = 0 \tag{77}$$

for the Rayleigh wave velocity c_{Rp}.

The dispersion relation c_R/c_s as a function of $\omega h/c_s$ is shown in Fig. 2 for Poisson's ratio $\nu = 1/3$, i.e. $k = 1/2$. It is seen that the plate edge Rayleigh wave velocity initially decreases somewhat when the frequency increases.

For $\Omega \ll 1$ one obtains:

$$P(C_R)+Q(C_R)\Omega^2+\Omega^4(\cdots) = 0, \tag{78}$$

where

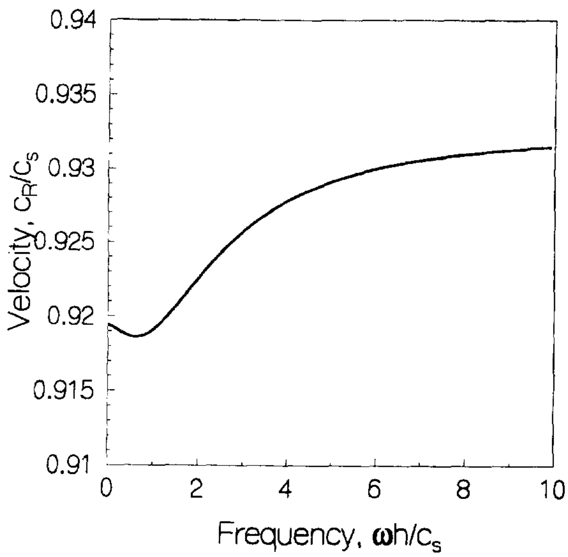

Fig. 2. Dispersion relation c_R/c_s vs $\omega h/c_s$ for Rayleigh waves. $c_R \to c_{\mathrm{Rpl}}$ as $\omega \to 0$, and $c_R \to c_{\mathrm{Rp}}$ as $\omega \to \infty$; $k = 1/2$ ($\nu = 1/3$).

$$P(C_R) = 4\sqrt{R_{pl}}\sqrt{R_s}-(2-C_R^2)^2 \tag{79}$$

$$Q(C_R) = \frac{1-C_R^2}{2C_R^2}\left\{(3-2C_R^2)(2-C_R^2)^2-8\sqrt{R_s}\sqrt{R_{pl}}\right.$$

$$\left.+8R_{pl}\frac{\sqrt{R_{pl}}}{\sqrt{R_s}}-4R_p\frac{\sqrt{R_p}}{\sqrt{R_{pl}}}\right\} \tag{80}$$

$$R_s = 1-C_R^2, \quad R_{pl} = 1-\kappa^2 C_R^2, \quad R_p = 1-k^2 C_R^2. \tag{81}$$

Function P vanishes for $C_R = C_{Rpl}$, the Rayleigh wave velocity in the plane stress approximation. A solution to eqn (78) within $O(\Omega^2)$ can be found by putting

$$C_R \approx C_{Rpl}+D\Omega^2, \tag{82}$$

where

$$D = -\frac{Q(C_{Rpl})}{P'(C_{Rpl})} \tag{83}$$

$$P'(C_{Rpl}) = 4C_{Rpl}\left\{(2-C_{Rpl}^2)-\frac{\kappa^2\sqrt{[R_s(C_{Rpl})]}}{\sqrt{[R_{pl}(C_{Rpl})]}}-\frac{\sqrt{[R_{pl}(C_{Rpl})]}}{\sqrt{[R_s(C_{Rpl})]}}\right\}. \tag{84}$$

One can show that $D < 0$, i.e. the velocity of Rayleigh waves along a plate edge decreases initially with increasing frequency, as observed from Fig. 2 for a special case. For this case, Poisson's ratio $\nu = 1/3$, i.e. $k = 1/2$, one obtains from eqn (82):

$$\frac{c_R}{c_s} \approx 0.91940-0.02393\left(\frac{h\omega}{c_s}\right)^2 \tag{85}$$

if $h\omega/c_s$ is small compared with unity.

6. CRACK PROPAGATION IN PLATES

Mode I crack propagation will be studied, since mode II is very seldom associated with crack propagation in plates. To simplify the investigation, the crack is assumed to be driven by forces on the crack faces, moving with constant velocity, V. Thus a wedging type loading is studied. The plate is assumed to be large enough, compared with the plate thickness and the load extension, that it can be regarded as infinite, and the crack is consequently regarded as semi-infinite, $x < Vt$, $y = 0$. With these assumptions, the Galilean transformation can be used:

$$x' = x-Vt, \quad y' = y, \quad z' = z, \tag{86}$$

where coordinates with a prime move with the crack edge. The transformation implies that

$$\frac{\partial F}{\partial t} = -V\frac{\partial F'}{\partial x'} \tag{87}$$

if $F'(x', y') = F(x, y, t)$.

For simplicity, the prime will be dropped in the following, i.e. x, y, z should be understood as the moving coordinates and previously used function symbols (such as Φ

and Ψ) as functions of these coordinates. The equations of motion, (36) and (37), will take the form

$$a_{\rm pl}^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}-\chi^2\left[a_{\rm s}^2a_{\rm p}^2\frac{\partial^4\Phi}{\partial x^4}-(a_{\rm s}^2+a_{\rm p}^2)\frac{\partial^4\Phi}{\partial x^2\,\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}\right]=0 \qquad (88)$$

$$a_{\rm s}^2\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=0, \qquad (89)$$

where

$$\chi=\frac{\kappa h}{\sqrt{3}} \qquad (90)$$

$$a_{\rm s}^2=1-\gamma^2,\quad a_{\rm s}>0,\quad a_{\rm pl}^2=1-\kappa^2\gamma^2,\quad a_{\rm pl}>0,\quad a_{\rm p}^2=1-k^2\gamma^2,\quad a_{\rm p}>0 \qquad (91)$$

and

$$\gamma=\frac{V}{c_{\rm s}}. \qquad (92)$$

The representations of the strains ε_x, ε_y and γ_{xy} will take the same form as eqns (38)–(40), whereas the strain ε_z and the normal stresses will be represented as

$$\varepsilon_z=-\frac{1}{1-2k^2}\left(a_{\rm p}^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right) \qquad (93)$$

$$\sigma_x=\mu\left[(1-a_{\rm s}^2)\frac{\partial^2\Phi}{\partial x^2}-2\frac{\partial^2\Phi}{\partial y^2}+2\frac{\partial^2\Psi}{\partial x\,\partial y}\right] \qquad (94)$$

$$\sigma_y=-\mu\left[(1+a_{\rm s}^2)\frac{\partial^2\Phi}{\partial x^2}+2\frac{\partial^2\Psi}{\partial x\,\partial y}\right] \qquad (95)$$

$$\sigma_z=-\frac{\mu(1-2\kappa^2)}{2}\left[a_{\rm pl}^2\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]. \qquad (96)$$

6.1. *Solution of the equations of motion*

With the ansatz

$$\Phi={\rm e}^{{\rm i}\alpha x+\lambda y} \qquad (97)$$

to the solution of eqn (88), one finds the characteristic equation

$$\lambda^4-\left[\alpha^2(a_{\rm s}^2+a_{\rm p}^2)+\frac{1}{\chi^2}\right]\lambda^2+\alpha^2\left(a_{\rm s}^2a_{\rm p}^2\alpha^2+\frac{a_{\rm pl}^2}{\chi^2}\right)=0 \qquad (98)$$

with the positive solutions

$$\lambda_1 = \sqrt{[r_1 - (r_1^2 - r_2)^{1/2}]}, \quad \lambda_2 = \sqrt{[r_1 + (r_1^2 - r_2)^{1/2}]} \tag{99}$$

$$r_1 = \left[\alpha^2(a_s^2 + a_p^2) + \frac{1}{\chi^2}\right] \Big/ 2 \quad r_2 = \alpha^2\left(a_s^2 a_p^2 \alpha^2 + \frac{a_{pl}^2}{\chi^2}\right). \tag{100}$$

When $\alpha \to 0$, $\lambda_1 \to a_{pl}\alpha$ and $\lambda_2 \to 1/\chi$.

Equation (89) is satisfied by

$$\Psi = e^{i\alpha x + a_s \alpha y}. \tag{101}$$

A general solution, which remains bounded as $y \to -\infty$, symmetrical with respect to x for Φ and anti-symmetrical for Ψ, is

$$\Phi = \int_0^\infty (A\,e^{\lambda_1 y} + B\,e^{\lambda_2 y}) \cos(\alpha x)\,d\alpha \tag{102}$$

$$\Psi = \int_0^\infty C\,e^{a_s \alpha y} \sin(\alpha x)\,d\alpha. \tag{103}$$

6.2. *Response to a load moving along the plate edge*

The edge $y = 0$ of the semi-infinite plate $y \leqslant 0$ is assumed to be subjected to a load, moving with velocity V in positive x direction, with the distribution

$$\sigma_y = -\frac{P}{\pi} \cdot \frac{\delta}{\delta^2 + x^2} \tag{104}$$

$$\tau_{xy} = \mu \gamma_{xy} = 0 \tag{105}$$

$$\tau_{yz} = \mu \frac{\partial \varepsilon_z}{\partial y} = 0. \tag{106}$$

These equations constitute the boundary conditions. When $\delta \to 0$ the right part of eqn (104) can be written as $-P$ times the Dirac delta function, i.e. the load is concentrated to a line load at $x = 0$. Insertion of eqns (102) and (103) into the boundary conditions, after use of eqns (95), (40) and (93), gives:

$$\int_0^\infty \alpha^2[(1 + a_s^2)(A + B) - 2a_s C] \cos(\alpha x)\,d\alpha = -\frac{P}{\pi\mu} \cdot \frac{\delta}{\delta^2 + x^2} \tag{107}$$

$$\int_0^\infty \alpha[2(\lambda_1 A + \lambda_2 B) - \alpha(1 + a_s^2)C] \sin(\alpha x)\,d\alpha = 0 \tag{108}$$

$$\int_0^\infty [\lambda_1(\lambda_1^2 - a_p^2\alpha^2)A + \lambda_2(\lambda_2^2 - a_p^2\alpha^2)B] \sin(\alpha x)\,d\alpha = 0. \tag{109}$$

Inversion gives

$$(1 + a_s^2)(A + B) - 2a_s C = -\frac{P\,e^{-\delta\alpha}}{\pi\mu\alpha^2} \tag{110}$$

$$2(\lambda_1 A + \lambda_2 B) - (1 + a_s^2)\alpha C = 0 \tag{111}$$

$$(\lambda_1^2 - a_p^2\alpha^2)\lambda_1 A + (\lambda_2^2 - a_p^2\alpha^2)\lambda_2 B = 0. \tag{112}$$

From these equations A, B and C are determined:

$$A = \frac{P\,\mathrm{e}^{-\delta\alpha}}{\pi\mu\alpha} \cdot \frac{(1+a_s^2)(\lambda_2^2 - a_p^2\alpha^2)\lambda_2}{(\lambda_2 - \lambda_1)D} \tag{113}$$

$$B = -\frac{P\,\mathrm{e}^{-\delta\alpha}}{\pi\mu\alpha} \cdot \frac{(1+a_s^2)(\lambda_1^2 - a_p^2\alpha^2)\lambda_1}{(\lambda_2 - \lambda_1)D} \tag{114}$$

$$C = \frac{P\,\mathrm{e}^{-\delta\alpha}}{\pi\mu\alpha^2} \cdot \frac{2(\lambda_1 + \lambda_2)\lambda_1\lambda_2}{D}, \tag{115}$$

where

$$D = 4a_s(\lambda_1 + \lambda_2)\lambda_1\lambda_2 - (1+a_s^2)^2\left[\lambda_1\lambda_2 + a_s^2\alpha^2 + \frac{1}{\chi^2}\right]\alpha. \tag{116}$$

The displacement gradient $\partial v/\partial x$ on $y = 0$ can now be determined. One obtains:

$$\frac{\partial v}{\partial x} = \frac{\partial^2\Phi}{\partial x\,\partial y} - \frac{\partial^2\Psi}{\partial x^2} = -\int_0^\infty \frac{1-a_s^2}{2}\alpha^2 C \sin(\alpha x)\,\mathrm{d}\alpha$$

$$= -\frac{(1-a_s^2)P}{\pi\mu}\int_0^\infty \mathrm{e}^{-\delta\alpha}K(\alpha)\sin(\alpha x)\,\mathrm{d}\alpha, \tag{117}$$

where

$$K(\alpha) = \frac{(\lambda_1 + \lambda_2)\lambda_1\lambda_2}{D}. \tag{118}$$

Asymptotic expressions for $K(\alpha)$ are:

$$K(\alpha) = \frac{a_{pl}}{4a_s a_{pl} - (1+a_s^2)^2} - \frac{\chi^2}{2} \cdot \frac{(1+a_s^2)^2(a_s^2 + a_{pl}^2)(a_p^2 - a_{pl}^2)}{a_{pl}[4a_s a_{pl} - (1+a_s^2)^2]^2}\alpha^2 + \cdots \tag{119}$$

$$K(\alpha) = \frac{a_p}{4a_s a_p - (1+a_s^2)^2} + \frac{1}{2\chi^2} \cdot \frac{(1+a_s^2)^2(a_p^2 - a_{pl}^2)(2a_p + a_s)}{a_s a_p (a_p + a_s)^2[4a_s a_{pl} - (1+a_s^2)^2]^2} \cdot \frac{1}{\alpha^2} + \cdots. \tag{120}$$

One observes that $K(0)$ is positive if $V < c_{Rpl}$, the Rayleigh wave velocity along a plate edge in the plane stress approximation, but negative if $V > c_{Rpl}$, and that $K(\infty)$ is positive if $V < c_{Rp}$, the plane strain Rayleigh velocity, but negative if $V > c_{Rp}$.

The integral in eqn (117) is convergent even for $\delta = 0$ at non-zero χ (i.e. with exception for the plane stress approximation), which implies that the response for a delta function load, $\sigma_y = -P\delta(x)$, is

$$\frac{\partial v}{\partial x} = -\frac{(1-a_s^2)P}{\pi\mu}\int_0^\infty K(\alpha)\sin(\alpha x)\,d\alpha. \tag{121}$$

If $\chi = 0$, i.e. the plane stress approximation, $K(\alpha) = K(0)$, and δ can be put to zero only after performing the integration. One obtains

$$\frac{\partial v}{\partial x} = -\frac{a_{pl}(1-a_s^2)P}{\pi\mu[4a_s a_{pl}-(1+a_s^2)^2]}\cdot\frac{1}{x}. \tag{122}$$

For plane strain one obtains the same result, except that a_{pl} is substituted by a_p. Therefore the solution to any problem associated with loads moving with constant velocity in plane stress or plane strain, for instance a train of coplanar cracks, can be obtained by solving the static problem ($V = 0$) and then multiplying by a velocity dependent function,

$$Y(\gamma) = \begin{cases} \dfrac{2(1-\kappa^2)a_{pl}(1-a_s^2)}{4a_s a_{pl}-(1+a_s^2)^2} & \text{for plane stress} \\ \dfrac{2(1-k^2)a_p(1-a_s^2)}{4a_s a_p-(1+a_s^2)^2} & \text{for plane strain} \end{cases}. \tag{123}$$

As is evident from the form of eqn (121), this is not possible in the present thin plate approximation. Function $Y(\gamma)$ could, appropriately, be called the Yoffe function, cf. Yoffe (1951).

6.3. *The crack problem*

The load on the crack faces is assumed to be

$$\sigma_y = f_0(x) \quad \text{for} \quad x < 0, y = 0. \tag{124}$$

It is assumed that $(\sqrt{x})f_0(x)$ is integrable; this condition becomes apparent later, after inspection of an analytic solution.

The semi-infinite plate $y \leqslant 0$ is regarded. The stresses on the boundary $y = 0$ are

$$\sigma_y = f_1(x) = f_0(x) + f(x) \tag{125}$$

$$\tau_{xy} = 0 \tag{126}$$

$$\tau_{yz} = 0, \tag{127}$$

where

$$f_0(x) = 0 \quad \text{for} \quad x > 0 \tag{128}$$

$$f(x) = 0 \quad \text{for} \quad x < 0. \tag{129}$$

$f(x)$ is thus unknown for $x > 0$, whereas $f_0(x)$ is known.

Now the function

$$g_1(x) = \frac{\pi\mu}{1-a_s^2}\left(\frac{\partial v}{\partial x}\right)_{y=0} \tag{130}$$

is introduced. It will be decomposed into two parts:

$$g_1(x) = g(x) + g_0(x), \tag{131}$$

where

$$g_0(x) = 0 \quad \text{for all} \quad x \tag{132}$$

$$g(x) = 0 \quad \text{for} \quad x > 0. \tag{133}$$

$g(x)$ is thus unknown for $x < 0$.

From the response (121) to $\sigma_y = -P\delta(x)$ one can obtain the response for the load distribution (125). This can be written as a convolution integral,

$$\sigma_y = f_1(x) = \int_{-\infty}^{+\infty} f_1(\xi)\delta(x-\xi)\,\mathrm{d}\xi \tag{134}$$

and, consequently, one obtains for $y = 0$:

$$g_1(x) = g(x) = \int_{-\infty}^{+\infty} [f(\xi)+f_0(\xi)] \int_0^{\infty} K(\alpha)\sin[\alpha(x-\xi)]\,\mathrm{d}\alpha\,\mathrm{d}\xi. \tag{135}$$

The following Laplace transforms are now introduced:

$$F(p) = p\int_{-\infty}^{+\infty} \mathrm{e}^{-px}f(x)\,\mathrm{d}x = p\int_0^{\infty} \mathrm{e}^{-px}f(x)\,\mathrm{d}x, \quad \mathscr{R}p \geqslant 0 \tag{136}$$

$$F_0(p) = p\int_{-\infty}^{+\infty} \mathrm{e}^{-px}f_0(x)\,\mathrm{d}x = p\int_{-\infty}^{0} \mathrm{e}^{-px}f_0(x)\,\mathrm{d}x, \quad \mathscr{R}p \leqslant 0 \tag{137}$$

$$G(p) = p\int_{-\infty}^{+\infty} \mathrm{e}^{-px}g(x)\,\mathrm{d}x = p\int_{-\infty}^{0} \mathrm{e}^{-px}g(x)\,\mathrm{d}x, \quad \mathscr{R}p \leqslant 0, \tag{138}$$

where $\mathscr{R}$ denotes the real part.

It should be noted that the transforms are defined in a dimension-true way. Since the right part of eqn (135) contains a convolution intergral, Laplace transformation of this equation encounters the transform

$$p\int_{-\infty}^{\infty} \mathrm{e}^{-px}\sin(\alpha x)\,\mathrm{d}x = \pi\alpha[\delta(\mathrm{i}p+\alpha)+\delta(\mathrm{i}p-\alpha)], \quad \mathscr{R}p = 0 \tag{139}$$

and thus the result is

$$\pi[F(p)+F_0(p)]\cdot\frac{|p|}{p}\cdot K(|p|) = G(p), \quad \mathscr{R}p = 0. \tag{140}$$

This is a Wiener-Hopf equation. $F(p)$ and $G(p)$, regular in the right and left p-half-plane, respectively, and on the imaginary axis, are to be determined, whereas $F_0(p)$, regular in the left p-half-plane, including the imaginary axis, is known. An investigation shows that $K(\alpha)$ possesses no real, non-negative singularities, and consequently $K(|p|)$ is regular on the imaginary axis.

In order to solve the equation, $K(|p|)$ will first be factorized. This can be done by decomposition of $\ln[K(|p|)/K(\infty)]$, which possesses the required behaviour as $p \to \pm\mathrm{i}\infty$. The result will be written as

$$K(|p|) = K(\infty)\cdot k_+(p)\cdot k_-(p), \quad \mathscr{R}p = 0, \tag{141}$$

where $k_+(p)$ is regular in the right p-half-plane, including the imaginary axis, and $k_-(p)$ is regular in the left p-half-plane, including the imaginary axis. One obtains:

$$\ln k_+(p) + \ln k_+(|p|) = \ln\frac{K(|p|)}{K(\infty)}, \quad \mathscr{R}p = 0 \tag{142}$$

$$\ln k_+(p) = -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\ln[K(|q|)/K(\infty)]}{q-p}\,dq, \quad \mathscr{R}p \geqslant 0 \tag{143}$$

$$\ln k_-(p) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\ln[K(|q|)/K(\infty)]}{q-p}\,dq, \quad \mathscr{R}p \leqslant 0. \tag{144}$$

When $p \to 0$ these integrals can be written as the sum of one integral taken in the sense of the Cauchy principal value (and vanishing, since the integrand is anti-symmetric), and one integral with the path indented around $p = 0$. The results are:

$$k_+(0) = k_-(0) = \sqrt{\frac{K(0)}{K(\infty)}}. \tag{145}$$

When $p \to \pm\infty$ the integrals are essentially determined by "large" values of the integration variable, and one obtains:

$$k_+(+\infty) = k_-(-\infty) = 1. \tag{146}$$

Next the factor $|p|/p$ in eqn (140) will be factorized. One obtains

$$\frac{|p|}{p} = \frac{r_+(p)}{r_-(p)}, \tag{147}$$

where

$$r_+(p) = p^{1/2}, \text{ branch cut along the negative real axis} \tag{148}$$

$$r_-(p) = ip^{1/2}, \text{ branch cut along the positive real axis.} \tag{149}$$

Branches are chosen so that r_+ is positive on the positive real axis and r_- is negative on the negative real axis.

Equation (140) can now be written in the form

$$F(p)r_+(p)k_+(p) + F_0(p)r_+(p)k_+(p) = \frac{r_-(p)}{\pi K(\infty)}\cdot\frac{G(p)}{k_-(p)}, \quad \mathscr{R}p = 0. \tag{150}$$

The second term in the left part will now be decomposed:

$$F_0(p)r_+(p)k_+(p) = p^2[L_+(p) + L_-(p)], \quad \mathscr{R}p = 0 \tag{151}$$

$$L_+(p) = -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{F_0(q)r_+(q)k_+(q)}{q^2(q-p)}\,dq, \quad \mathscr{R}p \geqslant 0 \tag{152}$$

$$L_-(p) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{F_0(q)r_+(q)k_+(q)}{q^2(q-p)}\,dq, \quad \mathscr{R}p \leqslant 0. \tag{153}$$

The integrals are convergent, since $k_+(p)$ is finite both as $p \to \pm i\infty$ and as $p \to 0$, $r_+(p)$ is proportional to $p^{1/2}$, and $F_0(p)$ is essentially proportional to p.

The Wiener-Hopf equation (150) can now be written so that the left part is regular for $\mathscr{R}p \geqslant 0$ and the right part regular for $\mathscr{R}p \leqslant 0$. Then they define together a function that is regular in the whole plane, and thus, since the physical quantities behave algebraically at infinity, this function must be a polynomial of finite degree. Thus, for $\mathscr{R}p = 0$,

$$F(p)r_+(p)k_+(p)+p^2L_+(p) = \frac{G(p)r_-(p)}{\pi K(\infty)k_-(p)} - p^2L_-(p)$$
$$= A_0 + A_1p + A_2p^2 + \cdots + A_np^n \tag{154}$$

and, consequently

$$F(p) = -\frac{p^2L_+(p)}{r_+(p)k_+(p)} + \frac{A_0 + A_1p + A_2p^2 + \cdots + A_np^n}{r_+(p)k_+(p)}, \quad \mathscr{R}p \geqslant 0 \tag{155}$$

$$G(p) = \pi\frac{K(\infty)k_-(p)p^2L_-(p)}{r_-(p)} + \pi K(\infty)k_-(p)\frac{A_0 + A_1p + A_2p^2 + \cdots + A_np^n}{r_-(p)}, \quad \mathscr{R}p \leqslant 0 \tag{156}$$

As $p \to +\infty$ the integral in the expression for $L_+(p)$ is essentially determined by "large" values of the integration variable, and thus

$$L_+(p) \to -\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{F_0(q)r_+(q)}{q^2(q-p)}\mathrm{d}q \quad \text{as} \quad p \to +\infty. \tag{157}$$

The path of integration can be deformed to a path following the lower side of the real axis from $q = -\infty$ to zero, encircling the branch point $q = 0$ of $r_+(q)$ and continuing along the upper side of the real axis to $q = -\infty$. After a subsequent change of integration variable, $q \to -q$, one obtains:

$$L_+(p) \to -\frac{1}{\pi}\int_0^{\infty}\frac{F_0(-q)}{q(\sqrt{q})(q+p)}\mathrm{d}q \quad \text{as} \quad p \to +\infty. \tag{158}$$

Similarly one obtains

$$L_-(p) \to \frac{1}{\pi}\int_0^{\infty}\frac{F_0(-q)}{q(\sqrt{q})(q+p)}\mathrm{d}q \quad \text{as} \quad p \to -\infty. \tag{159}$$

When $p \to +0$, the integral in the expression for $L_+(p)$ is essentially determined by "small" values of the integration variable and, in essentially the same way as for $p \to \pm\infty$, one obtains:

$$L_+(p) \to -\frac{1}{\pi}\sqrt{\frac{K(0)}{K(\infty)}}\int_0^\infty \frac{F_0(-q)}{q(\sqrt{q})(q+p)}\mathrm{d}q \quad \text{as} \quad p \to +0 \tag{160}$$

$$L_-(p) \to \frac{1}{\pi}\sqrt{\frac{K(0)}{K(\infty)}}\int_0^\infty \frac{F_0(-q)}{q(\sqrt{q})(q+p)}\mathrm{d}q \quad \text{as} \quad p \to -0. \tag{161}$$

Physical conditions now determine all constants A_0, $A_1 \dots A_n$ to 0. Since v is finite as $x \to -0$, $G(p)/p$ must be finite as $p \to -\infty$, which gives $A_2 = A_3 = \dots = A_n = 0$, and the condition that $f(x)$ is integrable implies that $F(p)/p$ must be finite as $p \to +0$, which gives $A_0 = A_1 = 0$.

It now remains to invert the transforms

$$\begin{aligned} F(p) &= -\frac{p^2 L_+(p)}{r_+(p)k_+(p)} \\ &= \frac{p\sqrt{p}}{2\pi i k_+(p)}\int_{-i\infty}^{+i\infty} \frac{F_0(q)r_+(q)k_+(q)}{q^2(q-p)}\mathrm{d}q, \quad \mathcal{R}p \geqslant 0 \end{aligned} \tag{162}$$

$$\begin{aligned} G(p) &= -\pi\frac{K(\infty)k_-(p)p^2 L_-(p)}{r_-(p)} \\ &= -\frac{K(\infty)k_-(p)p\sqrt{-p}}{2i}\int_{-i\infty}^{+i\infty} \frac{F_0(q)r_+(q)k_+(q)}{q^2(q-p)}\mathrm{d}q, \quad \mathcal{R}p \leqslant 0, \end{aligned} \tag{163}$$

where the last members in the two expressions are written in a form suitable for real values of p.

Functions $k_+(p)$ and $k_-(p)$ seem to be too complex to enable a full inversion; they will therefore be inverted only for values of x that are large, compared with the extension of the load $f_0(x)$ and to the plate thickness $2h$, and for values of x that are small, compared with these lengths. This corresponds to seeking asymptotic expressions for $F(p)$ and $G(p)$ as $p \to 0$ and $p \to \infty$, respectively.

As $p \to +0$ the integral in eqn (162) is determined by "small" values of the integration variable, and thus

$$F(p) \to \frac{p\sqrt{p}}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{F_0(q)r_+(q)}{q^2(q-p)}\mathrm{d}q \quad \text{as} \quad p \to +0. \tag{164}$$

In the same way as for $L_+(p)$ the path of integration is deformed towards the negative real axis and, after subsequent change of the sign of the integration variable, one obtains:

$$F(p) \to -\frac{p\sqrt{p}}{\pi}\int_0^\infty \frac{F_0(-q)}{q(\sqrt{q})(q+p)}\mathrm{d}q \quad \text{as} \quad p \to +0. \tag{165}$$

This transform can be inverted (see Appendix A), and one obtains:

$$f(x) \to -\frac{1}{\pi\sqrt{x}}\int_0^\infty \frac{(\sqrt{\xi})f_0(-\xi)}{x+\xi}\mathrm{d}\xi \quad \text{as} \quad x \to +\infty. \tag{166}$$

In the same way one finds the asymptotic expression for the strain gradient,

$$G(p) \to K(0)p\sqrt{-p}\int_0^\infty \frac{F_0(-q)}{q(\sqrt{q})(q+p)}\,\mathrm{d}q \quad \text{as} \quad p \to +0 \tag{167}$$

$$\frac{\partial v}{\partial x} \to \frac{a_{\mathrm{pl}}(1-a_s^2)}{\pi\mu[4a_s a_{\mathrm{pl}}-(1+a_s^2)^2]}\cdot\frac{1}{\sqrt{-x}}\int_0^\infty \frac{(\sqrt{\xi})f_0(-\xi)}{\xi - x}\,\mathrm{d}\xi \quad \text{as} \quad x \to -\infty. \tag{168}$$

Both asymptotic solutions for $|x| \to \infty$ agree with those obtained in the plane stress approximation, cf. Craggs (1960).

For "small" values of x it is suitable to make some variable substitutions in order to write $F(p)$ in the form

$$F(p) = \frac{1}{p}\cdot\frac{p}{k_+(p)}\cdot M(p), \tag{169}$$

where

$$M(p) = \frac{p\sqrt{p}}{\pi}\int_0^\infty \frac{A(s)p - B(s)s}{s(\sqrt{s})(s^2+p^2)}\,\mathrm{d}s \tag{170}$$

and, with $\mathscr{I}$ denoting the imaginary part,

$$A(s) = \mathscr{R}[F_0(\mathrm{i}s)k_+(\mathrm{i}s)\mathrm{e}^{\mathrm{i}\pi/4}] \tag{171}$$

$$B(s) = \mathscr{I}[F_0(\mathrm{i}s)k_+(\mathrm{i}s)\mathrm{e}^{\mathrm{i}\pi/4}]. \tag{172}$$

Inversion of $F(p)$ for "small" values of x is discussed in Appendix B. The result is

$$f(x) \to \frac{1}{\pi\sqrt{\pi}}\cdot\frac{1}{\sqrt{x}}\int_0^\infty \frac{A(s)}{s\sqrt{s}}\,\mathrm{d}s \quad \text{as} \quad x \to +0. \tag{173}$$

The asymptotic expression for the strain gradient as $x \to -0$ is obtained in essentially the same way:

$$\frac{\partial v}{\partial x} \to -\frac{a_p(1-a_s^2)}{\pi(\sqrt{\pi})\mu[4a_s a_p-(1+a_s^2)^2]}\cdot\frac{1}{\sqrt{-x}}\int_0^\infty \frac{A(s)}{s\sqrt{s}}\,\mathrm{d}s \quad \text{as} \quad x \to -0 \tag{174}$$

One can write [note that $k_+(\mathrm{i}s)$ is real]:

$$A(s) = -\frac{sk_+(\mathrm{i}s)}{\sqrt{2}}\int_{-\infty}^0 (\sin s|\xi| + \cos s|\xi|)f_0(\xi)\,\mathrm{d}\xi. \tag{175}$$

Thus

$$\int_0^\infty \frac{A(s)}{s\sqrt{s}}\,\mathrm{d}s = -\int_{-\infty}^0 f_0(\xi)\int_0^\infty \frac{k_+(\mathrm{i}s)}{\sqrt{(2s)}}(\sin s|\xi| + \cos s|\xi|)\,\mathrm{d}s\,\mathrm{d}\xi. \tag{176}$$

Assume now that the load $f_0(x)$ extends from $x = 0$ to $x = -L$ and that L is "very small"; this requirement will be discussed later. By writing the inner integral in eqn (176) as

$$\frac{1}{|\xi|}\int_0^\infty \frac{k_+(\mathrm{i}r/|\xi|)}{\sqrt{(2r)}}(\sin r+\cos r)\,\mathrm{d}r \tag{177}$$

and then, since only small values of $|\xi|$ are concerned, substituting $k_+(\mathrm{i}r/|\xi|)$ by $k_+(\infty) = 1$, one obtains:

$$\int_0^\infty \frac{A(s)}{s\sqrt{s}}\,\mathrm{d}s = -\sqrt{\pi}\int_{-\infty}^0 \frac{f_0(\xi)}{\sqrt{-\xi}}\,\mathrm{d}\xi \tag{178}$$

and, consequently,

$$f(x) \to -\frac{1}{\pi}\cdot\frac{1}{\sqrt{x}}\int_0^\infty \frac{f_0(-\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi \quad \text{as} \quad x\to +0 \tag{179}$$

$$\frac{\partial v}{\partial x} \to -\frac{a_\mathrm{p}(1-a_\mathrm{s}^2)}{\pi\mu[4a_\mathrm{s}a_\mathrm{p}-(1+a_\mathrm{s}^2)^2]}\cdot\frac{1}{\sqrt{-x}}\int_0^\infty \frac{f_0(-\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi \quad \text{as} \quad x\to -0. \tag{180}$$

This result is actually *the plane strain solution*, cf. Craggs (1960), for the crack edge neighbourhood. On the other hand, if L is "very large" (to be discussed later), one can substitute $k_+(\mathrm{i}r/|\xi|)$ by $k_+(0) = \sqrt{[K(0)/K(\infty)]}$ and one obtains:

$$f(x) \to -\sqrt{\left\{\frac{a_\mathrm{pl}[4a_\mathrm{s}a_\mathrm{p}-(1+a_\mathrm{s}^2)^2]}{a_\mathrm{p}[4a_\mathrm{s}a_\mathrm{pl}-(1+a_\mathrm{s}^2)^2]}\right\}}\cdot\frac{1}{\pi}\cdot\frac{1}{\sqrt{x}}\int_0^\infty \frac{f_0(-\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{181}$$

$$\frac{\partial v}{\partial x} \to -\frac{\sqrt{(a_\mathrm{p}a_\mathrm{pl})}(1-a_\mathrm{s}^2)}{\pi\mu\sqrt{\{[4a_\mathrm{s}a_\mathrm{p}-(1+a_\mathrm{s}^2)^2][4a_\mathrm{s}a_\mathrm{pl}-(1+a_\mathrm{s}^2)^2]\}}}\cdot\frac{1}{\sqrt{-x}}\int_0^\infty \frac{f_0(-\xi)}{\sqrt{\xi}}\,\mathrm{d}\xi \tag{182}$$

as $|x| \to 0$. This result can be described as *plane stress like* behaviour near the crack edge: one observes in particular that the energy release rate is the one found in the plane stress approximation, although the expressions for $f(x)$ and $\partial v/\partial x$ are such that the ratio between the stress intensity factor and the corresponding factor for $\partial v/\partial x$ coincides with the ratio characteristic of plane strain.

As to the question about the meaning of "very small" and "very large" values of L, one notices that the function $k_+(\mathrm{i}r/|\xi|)$ in the integral

$$\frac{1}{|\xi|}\int_0^\infty \frac{k_+(\mathrm{i}r/|\xi|)}{\sqrt{(2r)}}(\sin r+\cos r)\,\mathrm{d}r \tag{183}$$

decreases from $k_+(0)$ to $k_+(\infty)$ as r increases, and that, as follows from eqn (119) the essential part of this decrease takes place during the interval from $r/|\xi| = 0$ to

$$\frac{r}{|\xi|} \approx \frac{2a_\mathrm{pl}}{\chi(1+a_\mathrm{s}^2)}\sqrt{\left(\frac{4a_\mathrm{s}a_\mathrm{pl}-(1+a_\mathrm{s}^2)^2}{(a_\mathrm{s}^2+a_\mathrm{pl}^2)(a_\mathrm{p}^2-a_\mathrm{pl}^2)}\right)} = \frac{1}{R(\gamma)\chi}. \tag{184}$$

One further notices that the integral

$$\int_0^{r_\infty} \frac{\sin r + \cos r}{\sqrt{(2r)}} \mathrm{d}r \tag{185}$$

equals $\sqrt{\pi}$ if $r_\infty = \infty$, and deviates less than 20% from this value if $r_\infty \geqslant 2\pi$. The inner integral in eqn (176) is thus essentially determined by large values of $s = r/|\xi|$ if

$$\frac{L}{\chi} \ll 2\pi R(\gamma) \tag{186}$$

which can be taken as a criterion for plane strain domination of the field near the crack edge. Small values of $r/|\xi|$ give the major contribution to the integral if

$$\frac{L}{\chi} \gg 2\pi R(\gamma) \tag{187}$$

which can be taken as a criterion for a plane stress like field near the crack edge.

For values of k around 0.5 (ν around 0.33) the function $R(\gamma)$ is virtually independent of γ up to about 95% of the Rayleigh wave velocity in the plane stress approximation. It equals 1/2 for $k = 1/2$ and $\gamma = 0$. The quantity $2\pi R(\gamma)$ is therefore approximately equal to 3 for most interesting applications. Since $\chi = h/3$ for $k = 1/2$ the transition from a plane strain dominated field near the crack edge to a plane stress like field will take place when the extension L of the load becomes of the same order as the plate half-thickness h.

7. DISCUSSION

Applied to cases in which the crack faces are traction free the results indicate that plane stress like behaviour near the crack edge occurs for cracks that are considerably longer than the plate thickness, at least for velocities that are not close to the Rayleigh wave velocity in the plane stress approximation. Since this is by far the most common situation, one can conclude that in general the field near the edge of a running crack in a plate shows plane stress like behaviour. Such behaviour is not identical with that found from the plane stress approximation. For the case $k = 1/2$ ($\nu = 1/3$) one finds that the stress intensity factor is about 6% higher than in the plane stress approximation, and the crack opening displacement about 6% lower. This departure from the plane stress approximation indicates the plane strain infuence near the crack edge: one can envisage a field rather similar to that in the plane stress approximation beyond some distance from the crack edge, but the plane strain contribution in the close vicinity of the edge acts stiffening on the stress–strain field. The plane strain influence near the crack edge is manifested by the fact that the ratio between the stress intensity factor and the corresponding factor for $\partial v/\partial x$ coincides with the ratio characteristic of plane strain.

By studying the behaviour of ε_z near the crack edge it might perhaps be possible to estimate how far from the edge the field agrees approximately with the field in the plane stress approximation. The results obtained about the influence of the extension L of the load $f(x)$ might perhaps be interpreted so that the plane stress approximation is fairly well realized at distances from the crack edge that are of the same order as half the plate thickness or larger. This can be compared with the estimates by Yang and Freund (1985) and by Rosakis and Ravi-Chandar (1986) for stationary cracks, that the plane stress approximation is approximately valid at distances larger than half the plate thickness from the crack edge.

The energy release rate at plane stress like behaviour near the crack edge was found to be the same as in the plane stress approximation, and therefore, as it appears, the same as it would be if the stress intensity factor and the crack opening displacement were determined at a somewhat larger distance from the crack edge. This result can be compared with the result for stationary cracks in a plate where the J integral for a path near the crack edge equals the J integral for a sufficiently remote path (Broberg, 1987).

It is not easy to extract the influence of lateral inertia from other effects, particularly the one of apparent plane strain conditions near the crack edge, but it is clearly of minor importance. This result is not obvious: an attempt to estimate the energy due to lateral motion from the displacement w calculated by using the plane stress approximation, indicates a considerable importance of the lateral inertia.

The theoretical maximum crack edge velocity appears to be that of plane stress approximation Rayleigh waves, even though the plane strain Rayleigh wave velocity could be the theoretical maximum in the rare event that the extension L of the load $f(x)$ is considerably shorter than the plate half-thickness.

The present approximation for crack propagation in a plate does not ensure traction-free plate surfaces. This could be achieved by going one step further and allowing a parabolic rather than a constant variation over the thickness of the stress ε_z. This possibility is outlined in Appendix C. It is believed, however, that the main conclusions in the present paper remain valid, at least approximately, even in such a more accurate approximation.

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APPENDIX A

The expression

$$F(p) = \frac{p\sqrt{p}}{\pi}\int_0^\infty \frac{F_0(-q)}{q(\sqrt{q})(q+p)}\,\mathrm{d}q \tag{A1}$$

which occurs in eqn (165), is to be inverted. Insertion of the expression for $F_0(q)$ gives

$$F(p) = -\frac{p\sqrt{p}}{\pi}\int_0^\infty\int_0^\infty \mathrm{e}^{-q\xi}f_0(-\xi)\,\mathrm{d}\xi\frac{\mathrm{d}q}{(\sqrt{q})(q+p)}. \tag{A2}$$

If the Laplace transform of a function of x is written with the operator L_{px} and the inverse operator as L_{px}^{-1}, one obtains:

$$\begin{aligned} f(x) &= -\frac{1}{\pi}L_{px}^{-1}\left\{p\sqrt{p}\int_0^\infty\int_0^\infty \mathrm{e}^{-q\xi}f_0(-\xi)\,\mathrm{d}\xi\frac{\mathrm{d}q}{(\sqrt{q})(q+p)}\right\} \\ &= -\frac{1}{\pi}\int_0^\infty\frac{f_0(-\xi)}{\xi}L_{px}^{-1}\left\{\xi\int_0^\infty\frac{\mathrm{e}^{-\xi q}p\sqrt{p}}{(\sqrt{q})(q+p)}\right\}\mathrm{d}\xi \\ &= -\frac{1}{\pi}\int_0^\infty\frac{f_0(-\xi)}{\xi}L_{px}^{-1}\left\{L_{\xi q}\frac{1}{\left(\sqrt{\frac{q}{p}}\right)\left(\frac{q}{p}+1\right)}\right\}\mathrm{d}\xi; \end{aligned} \tag{A3}$$

but

$$L_{\xi q}\frac{1}{\left(\sqrt{\frac{q}{p}}\right)\left(\frac{q}{p}+1\right)} \tag{A4}$$

is a function of $p\xi$, say

$$L_{\xi q}\frac{1}{\left(\sqrt{\frac{q}{p}}\right)\left(\frac{q}{p}+1\right)} = h(p\xi) = h(\xi p) = L_{px}\frac{1}{\left(\sqrt{\frac{x}{\xi}}\right)\left(\frac{x}{\xi}+1\right)}. \tag{A5}$$

Thus

$$L_{px}^{-1}\left\{L_{\xi q}\frac{1}{\left(\sqrt{\frac{q}{p}}\right)\left(\frac{q}{p}+1\right)}\right\}=\frac{1}{\left(\sqrt{\frac{x}{\xi}}\right)\left(\frac{x}{\xi}+1\right)} \tag{A6}$$

and one obtains:

$$f(x)=-\frac{1}{\pi\sqrt{x}}\int_0^\infty\frac{(\sqrt{\xi})f_0(-\xi)}{x+\xi}\,\mathrm{d}\xi. \tag{A7}$$

APPENDIX B

Inversion of the expression

$$\frac{p^2\sqrt{p}}{p^2+s^2} \tag{B1}$$

which appears in eqn (169), gives the result

$$\frac{1}{\sqrt{(\pi x)}}-\frac{1}{\sqrt{\pi}}\sqrt{s}\int_0^{sx}\frac{\sin(sx-u)}{\sqrt{u}}\,\mathrm{d}u,\quad x>0 \tag{B2}$$

and inversion of

$$\frac{p\sqrt{p}}{p^2+s^2} \tag{B3}$$

also appearing in eqn (169), gives

$$\frac{1}{\sqrt{(\pi s)}}\int_0^{sx}\frac{\cos(sx-u)}{\sqrt{u}}\,\mathrm{d}u,\quad x>0. \tag{B4}$$

Therefore the inversion of $M(p)$ is

$$m(x)=\frac{1}{\pi\sqrt{\pi}}\int_0^\infty\left\{\frac{A(s)}{s\sqrt{s}}\cdot\frac{1}{\sqrt{x}}-\frac{A(s)}{s}\int_0^{sx}\frac{\sin(sx-u)}{\sqrt{u}}\,\mathrm{d}u-\frac{B(s)}{s}\int_0^{sx}\frac{\cos(sx-u)}{\sqrt{u}}\,\mathrm{d}u\right\}\mathrm{d}s. \tag{B5}$$

For "small" values of x inversion of

$$\frac{p}{k_+(p)} \tag{B6}$$

appearing in eqn (169), is found by letting $p\to+\infty$. From eqn (143) one obtains:

$$\ln k_+(p)=\frac{p}{\pi}\int_0^\infty\frac{\ln[K(s)/K(\infty)]}{p^2+s^2}\,\mathrm{d}s \tag{B7}$$

and thus

$$\frac{p}{k_+(p)}\to p-\frac{1}{\pi}\int_0^\infty\ln\frac{K(s)}{K(\infty)}\quad\text{as}\quad p\to+\infty. \tag{B8}$$

Use of asymptotic relations for Laplace transforms gives

$$k(x)\to\delta(x)-\frac{1}{\pi}\int_0^\infty\ln\frac{K(s)}{K(\infty)}\,\mathrm{d}s\quad\text{as}\quad x\to+0, \tag{B9}$$

where $k(x)$ is the inversion of $p/k_+(p)$. Since the inversion of $F(p)$, given in eqn (169), is

$$\int_0^x k(\xi)m(x-\xi)\,\mathrm{d}\xi \tag{B10}$$

one obtains

$$f(x) \to \frac{1}{\pi\sqrt{\pi}} \cdot \frac{1}{\sqrt{x}} \int_0^\infty \frac{A(s)}{s\sqrt{s}} \mathrm{d}s \quad \text{as} \quad x \to +0. \tag{B11}$$

APPENDIX C

In the main text the strain ε_z was supposed to be constant over the thickness.

Here a parabolic variation is assumed. This implies that

$$u = u(x, y, t), \quad v = v(x, y, t) \tag{C1}$$

$$w = \varepsilon_0(x, y, t) \cdot z + \varepsilon_1(x, y, t) \cdot \frac{z^3}{h^2}. \tag{C2}$$

Then

$$\varepsilon_x = \varepsilon_x(x, y, t), \quad \varepsilon_y = \varepsilon_y(x, y, t) \tag{C3}$$

$$\varepsilon_z = \varepsilon_0(x, y, t) + 3\varepsilon_1(x, y, t) \cdot \frac{z^2}{h^2} \tag{C4}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{C5}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} = z\frac{\partial \varepsilon_0}{\partial y} + \frac{z^3}{h^2} \cdot \frac{\partial \varepsilon_1}{\partial y} \tag{C6}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} = z\frac{\partial \varepsilon_0}{\partial x} + \frac{z^3}{h^2} \cdot \frac{\partial \varepsilon_1}{\partial x}. \tag{C7}$$

The kinetic energy, T, and the potential energy, U, are given as:

$$T = \frac{\rho}{2}(\dot{u}^2 + \dot{v}^2 + \dot{w}^2) = \frac{\rho}{2}\left[\dot{u}^2 + \dot{v}^2 + \left(\dot{\varepsilon}_0 z + \dot{\varepsilon}_1 \frac{z^3}{h^2}\right)^2\right] \tag{C8}$$

$$U = \tfrac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}). \tag{C9}$$

The condition of traction-free plate faces

$$\sigma_z = 0 \quad \text{for} \quad z = \pm h \tag{C10}$$

can now be satisfied. One obtains, using Hooke's law:

$$\varepsilon_0 + 3\varepsilon_1 = -\frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_y). \tag{C11}$$

ε_1 can thus be expressed in terms of ε_0, ε_x and ε_y. Thus one can write

$$\int_{-h}^{h} (T-U)\,\mathrm{d}z = F\left(\dot{u}, \dot{v}, \dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_0, \varepsilon_x, \varepsilon_y, \varepsilon_0, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial \varepsilon_x}{\partial x}, \frac{\partial \varepsilon_x}{\partial y}, \frac{\partial \varepsilon_y}{\partial x}, \frac{\partial \varepsilon_y}{\partial y}, \frac{\partial \varepsilon_0}{\partial x}, \frac{\partial \varepsilon_0}{\partial y}\right) \tag{C12}$$

and the variation of

$$\delta I = \delta \int_t \int_x \int_y \int_{-h}^{h} (T-U)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t = 0 \tag{C13}$$

can be performed with respect to variations of u, v and ε_0, producing three equations of motion, corresponding to eqns (24)–(26).